Properties of Codes with the Rank Metric

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Abstract—In this paper, we study properties of rank metric codes in general and maximum rank distance (MRD) codes in particular. For codes with the rank metric, we first establish Gilbert and sphere-packing bounds, and then obtain the asymptotic forms of these two bounds and the Singleton bound. Based on the asymptotic bounds, we observe that asymptotically Gilbert-Varsharmov bound is exceeded by MRD codes and sphere-packing bound cannot be attained. We also establish bounds on the rank covering radius of maximal codes, and show that all MRD codes are maximal codes and all the MRD codes known so far achieve the maximum rank covering radius.

I. INTRODUCTION

Early in the development of coding theory, it was found convenient to model communication channels as conveyors of symbols from finite sets and represent the effects of channel noise by occasional reception of a symbol other than the transmitted symbol. Thus, the Hamming metric has been considered the most relevant metric for error-control codes. Recently, the rank metric has attracted some attention due to its relevance to wireless communications [1] and storage equipments [2]. In [3], space-time block codes with good rank properties have been proposed. Rank metric codes are used to correct crisscross errors that can be found in memory chip arrays and magnetic tapes [2]. Codes with the rank metric have also been used in the Gabidulin-Paramonov-Tretjakov (GPT) public-key cryptosystem [4] and its variants (see, for example, [5], [6]). The public-key cryptosystems based on codes with the rank metric have much smaller public key sizes than those for Hamming metric based public-key cryptosystems such as McEliece’s cryptosystem [7].

Due to these potential applications, general properties of codes with the rank metric have received some attention [2], [8]–[14]. The rank metric was considered as a metric for block codes over extension fields in [8], whereas in [2] the rank metric was considered for array codes that consist of arrays over base fields. However, in both [8] and [2], the same family of codes that are optimal in the metric sense were proposed. In [10], the rank distance properties of primitive length linear cyclic codes were studied. In [11], the rank covering radius of codes was studied, and the sphere-covering bound for the rank metric was introduced. The concept of rank covering radius was generalized in [12], where the multi-covering radii of codes with the rank metric were defined. Recently, a somewhat more general construction for MRD codes was proposed in [15], and the properties of subspace subcodes and subfield subcodes were considered in [16].

In this paper, we study properties of rank metric codes in general and MRD codes in particular. For codes with the rank metric, we first establish Gilbert and sphere-packing bounds, and then obtain the asymptotic forms of these two bounds and the Singleton bound. Based on these asymptotic bounds, we observe that MRD codes exceed the Gilbert-Varsharmov bound, and that asymptotically perfect codes (codes that attain the sphere-packing bound) do not exist. We also establish bounds on the rank covering radius of maximal codes, and show that all MRD codes are maximal codes and all the MRD codes known so far achieve the maximum rank covering radius. The number of vectors with certain weights determines the security against decoding attacks of public-key cryptosystems based on error-control codes. We compare the distributions of rank and Hamming weights of vectors, and use the difference to partially explain why GPT cryptosystem and its variants are quite secure against decoding attacks.

The rest of the paper is organized as follows. Section II reviews necessary backgrounids in an effort to make this paper self-contained. In Section III, we propose the Gilbert bound and the sphere-packing bound for codes with the rank metric and their asymptotic forms as well as the asymptotic form of the Singleton bound. Section IV establishes bounds on the rank covering radius of maximal codes, and shows that all MRD codes are maximal codes and all the MRD codes known so far achieve the maximum rank covering radius. In Section V, the distributions of rank and Hamming weights of vectors are compared and the difference partially explains why GPT cryptosystem and its variants are quite secure against decoding attacks.

II. PRELIMINARIES

A. Rank metric

Consider $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \text{GF}(q^n)$, the $n$-dimensional vector space over $\text{GF}(q^m)$. Assume $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$ is a basis set of $\text{GF}(q^m)$ over $\text{GF}(q)$, then for $j = 0, 1, \ldots, n-1$ $a_j$ can be written as $a_j = \sum_{i=0}^{m-1} a_{i,j} \gamma_i$, where $a_{i,j} \in \text{GF}(q)$ for $i = 0, 1, \ldots, m-1$. Hence, $a_j$ can be expanded to an $m$-dimensional column vector $(a_{0,j}, a_{1,j}, \ldots, a_{m-1,j})^T$ with respect to the basis set $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$. Let $A$ be the $m \times n$ matrix obtained by
expanding all the coordinates of \(a\), i.e.,
\[
A = \begin{pmatrix}
  a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\
  a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1}
\end{pmatrix},
\]
where \(a_j = \sum_{i=0}^{m-1} a_{ij} \gamma_i\). The rank norm (over \(GF(q^m)\)) of the vector \(a\), denoted as \(rk(a)\), is defined to be the rank of the matrix \(A\) over \(GF(q)\), i.e., \(rk(a) = rank(A)\) [8].

Accordingly, \(\forall a, b \in GF(q^m)^n\), \(d_a(a, b) \equiv rk(a - b)\) is shown to be a metric over \(GF(q^m)^n\), referred to as the rank metric henceforth [8]. Hence, the minimum rank distance \(d_a\) of a code is simply the minimum rank distance over all possible pairs of distinct codewords. Clearly, a code with a minimum rank distance \(d_a\) can correct errors with rank up to \(t = \left\lfloor (d_a - 1)/2 \right\rfloor\).

**B. The Singleton bound and MRD codes**

The minimum rank distance of a code can be specifically bounded. First, the minimum rank distance \(d_a\) of a code over \(GF(q^m)\) is obviously bounded by \(m\). Codes that satisfy \(d_a = m\) were referred to as full rank distance codes and were studied in [14]. Also, it can be shown that \(d_a \leq d_h\) [8], where \(d_h\) is the minimum Hamming distance of the same code. Due to the Singleton bound for block codes, the minimum rank distance of an \((n, k)\) block code over \(GF(q^m)\) thus satisfies
\[
d_h \leq n - k + 1. \tag{1}
\]

An alternative bound on the minimum rank distance is also given in [17]:
\[
d_a \leq \left\lfloor \frac{m}{n} (n - k) \right\rfloor + 1. \tag{2}
\]
For \(n \leq m\), the bound in (1) is tighter than that in (2). When \(n > m\) the bound in (2) is tighter. Since \(\left\lfloor \frac{m}{n} (n - k) \right\rfloor + 1 \leq m\) and the equality holds only when \(\frac{m}{n} (n - k) \leq 1\), the minimum rank distance of a code must satisfy:
\[
d_a \leq \min \left\{ n - k + 1, \left\lfloor \frac{m}{n} (n - k) \right\rfloor + 1 \right\}. \tag{3}
\]

In this paper, we refer to the bound in (3) as the Singleton bound\(^2\) for codes with the rank metric, and call codes that attain the bound as maximum rank distance (MRD) codes. MRD codes are the equivalent in the rank metric of maximum rank distance separable (MDS) codes [18] in the Hamming metric.

Three subclasses of MRD codes have been proposed to our best knowledge. The first subclass of MRD codes, called Gabidulin codes, was first introduced in [8].

**Definition 1 (Gabidulin codes):** When \(n \leq m\), let \(g = (g_0, g_1, \ldots, g_{n-1})\) be linearly independent elements of \(GF(q^m)\). Then the code defined by the following generator matrix
\[
G = \begin{pmatrix}
  g_0[0] & g_1[0] & \cdots & g_{n-1}[0] \\
  g_0[1] & g_1[1] & \cdots & g_{n-1}[1] \\
  \vdots & \vdots & \ddots & \vdots \\
  g_0[k-1] & g_1[k-1] & \cdots & g_{n-1}[k-1]
\end{pmatrix}, \tag{4}
\]
where \([i] = q^i\), is called a Gabidulin code, generated by \(g = (g_0, g_1, \ldots, g_{n-1})\), with dimension \(k\) and minimum rank distance \(d_a = n - k + 1\).

A second subclass of MRD codes, referred to as generalized Gabidulin codes henceforth, was recently introduced in [15]. These codes have a similar generator matrix to that in (4) except that for this subclass of codes \([i] = q^a\) with \(a\) being an integer prime to \(m\). Even though Gabidulin codes are only a special case of generalized Gabidulin codes (for \(a = 1\)), we consider Gabidulin codes as a separate subclass since their properties have been studied more extensively. The third subclass of MRD codes consists of cartesian products of an MRD code with length \(n = m\). Let \(C\) be an \((n, k, d_a = n - k + 1)\) MRD code over \(GF(q^m)\), and let \(C^l \equiv C \times \ldots \times C\) be the code obtained by \(l\) cartesian products of \(C\). Thus, \(C^l\) is an code with length \(n^l = nl\), dimension \(k^l = kl\), and minimum rank distance \(d_a^l = d_a = n - k + 1\). It can be shown that \(C^l\) is an MRD code and only if \(n = m\). Note that the first two subclasses of MRD codes have length less than \(m\), whereas this third subclass consists of codes with length \(n^l = lm \geq m\).

**III. BOUNDS FOR THE RANK METRIC**

**A. Gilbert and sphere-packing bounds for the rank metric**

Let us denote the number of \(n\)-dimensional vectors with weight \(w\) over \(GF(q^m)\) as \(N_{q^m}(n, w)\). For the rank metric, it is given by
\[
N_{q^m}(n, w) = \prod_{i=0}^{w-1} \frac{(q^m - q^i)(q^m - q^i - 1)}{(q^w - q^i)}. \tag{5}
\]
Denoting the volume of a ball of radius \(w\) (in the rank metric) in \(GF(q^m)^n\) by \(V_{q^m}(n, w)\), we obtain
\[
V_{q^m}(n, w) = \sum_{j=0}^{w} N_{q^m}(n, j). \tag{6}
\]

Bounds on the parameters of codes indicate how good codes are, and also provide guidelines to the design of good codes. The Gilbert and sphere-packing bounds are two important bounds for codes in the Hamming metric [18], [19]. First, the Gilbert bound states that there always exists a code with \(A\) codewords, length \(n\), minimum distance \(d\) such that \(A \geq V_{q^m}(n, d-1)\). The sphere-packing bound, on the other hand, states that any code has to satisfy \(A \leq \frac{q^m^n}{V_{q^m}(n, d)}\), where \(t = \left\lfloor (d - 1)/2 \right\rfloor\). The derivations of these two bounds are transparent to the metric considered, and hence these two bounds can be easily adapted to codes with the rank metric.
Let us denote the maximum number of codewords in a code of length $n$ and minimum rank distance $d_k$ over $\text{GF}(q^m)$ as $A_{q^m}(n, d_k)$. The Gilbert and sphere-packing bounds for codes with the rank metric are given by

$$
\frac{q^{mn}}{V_{q^m}(n, d_k - 1)} \leq A_{q^m}(n, d_k) \leq \frac{q^{mn}}{V_{q^m}(n, t)}.
$$

(7)

The Gilbert bound gives a lower bound to the cardinality of a “reasonably good” code for given block length and minimum distance. More formally, one can show that the Gilbert bound is always reached or exceeded by a special class of codes, called maximal codes [20].

**Definition 2 (Maximal code):** A code $C$ with length $n$ and minimum rank distance $d_k$ is maximal if there does not exist any code $C'$ with same length and minimum rank distance such that $C \subset C'$.

We can show that

**Proposition 1:** All MRD codes are maximal codes.

Due to limited space, all proofs are omitted in this paper and they will be presented at the conference. We emphasize that Proposition 1 applies to any MRD code, i.e., any code that maximizes the minimum rank distance for given length and cardinality. Obviously, the three known subclasses of MRD codes are all maximal codes. However, cartesian products of Gabidulin (or generalized Gabidulin) codes are not necessarily maximal codes. Indeed, we can show that

**Proposition 2:** Let $C$ be an $(n, k, d_k)$ MRD code over $\text{GF}(q^m)$ $(n \leq m)$, and let $C'$ be the $(n', k', d'_k)$ code obtained by $l$ cartesian products of $C$. If $l \geq \frac{m}{m-n}$ and $d_k > 1$, then $C'$ is not a maximal code.

**B. Asymptotic bounds for the rank metric**

The performance of codes of large block length can be studied in terms of asymptotic bounds on the relative minimum distance in the limit of infinite block length. In this section, we will study the asymptotic forms of the three bounds in (3) and (7) respectively in the case where both block length and minimum rank distance go to infinity. However, this cannot be achieved for finite $m$ since the minimum rank distance is no greater than $m$. Thus, we consider only the case where $\lim_{n \to \infty} \frac{n}{m} = b$, where $b$ is a constant.

Define

$$
\delta \overset{\text{def}}{=} \frac{d_k}{n} \quad \text{and} \quad a(\delta) \overset{\text{def}}{=} \lim_{n \to \infty} \sup \left[ \log_{q^m} A_{q^m}(n, \lfloor \delta n \rfloor) \right],
$$

where $a(\delta)$ represents the maximum possible code rate of a code which has relative minimum distance $\delta$ as its length goes to infinity. The asymptotic forms of the bounds in (7) are given in the propositions below.

**Proposition 3 (Gilbert-Varsharmov bound):** The asymptotic behavior of the Gilbert bound for the rank metric is given by (for $0 \leq \delta \leq \min\{1, b^{-1}\})

$$
a(\delta) \geq (1 - \delta)(1 - b\delta),
$$

(8)

which will be referred to as the Gilbert-Varsharmov bound for the rank metric.

**Proposition 4 (Asymptotic sphere-packing bound):** The asymptotic behavior of the sphere-packing bound for the rank metric is given by (for $0 \leq \delta \leq \min\{1, b^{-1}\})

$$
a(\delta) \leq \left(1 - \frac{\delta}{2}\right) \left(1 - b\frac{\delta}{2}\right).
$$

(9)

The Singleton bound for the rank metric in (3) asymptotically becomes

$$
a(\delta) \leq \left\{ \begin{array}{ll}
1 - \delta & \text{if } b \leq 1, \\
1 - b\delta & \text{if } b > 1.
\end{array} \right.
$$

(10)

The three asymptotic bounds are illustrated in Figures 1, 2, and 3 for $b = 1, 4$, and 0.25, respectively. Insights about asymptotic behavior of codes in the rank metric can be
obtained from these asymptotic bounds. First, note that for \( \delta > 0 \) the sphere-packing bound is always looser than the Singleton bound. For example, when \( b = 1 \) and \( \delta = 1 \), then the right hand side of (9) is 1/4, although \( a(\delta) = 0 \). Since both the sphere-packing and Singleton bounds are upper bounds, this implies that the sphere-packing bound cannot be attained asymptotically. That is, asymptotically there are no perfect codes in the rank metric. This confirms the claim in [9] that there are no perfect codes for the rank metric.

The values 0, 1, and \( \infty \) are the special cases for \( b \). When \( b = 0 \), the right hand sides of (8) and (10) coincide. This means that when \( m \) increases faster than linearly with \( n \), then MRD codes do not exceed the Gilbert-Varshamov bound. This is similar to the case where the Gilbert-Varshamov bound for codes with the Hamming metric is reached but not exceeded by MDS codes. When \( b = 1 \), it can be shown that the gap between the Singleton bound and Gilbert-Varshamov bound is maximized. When \( b \to \infty \), the bounds in (8), (9), and (10) are valid only at the point \( \delta = 0 \). This is because \( \lim_{n \to \infty} \frac{d_k}{n} \leq \lim_{n \to \infty} \frac{m}{n} = 0 \).

Since there exist MRD codes for any value of \( \delta \), the bound in (10) is attainable by MRD codes. We note that the asymptotic code rates of MRD codes are always greater than \((1-\delta)(1-b\delta)\) unless \( b = 0 \). Therefore, MRD codes exceed the Gilbert-Varshamov bound for the rank metric unless \( b = 0 \), as illustrated in Figures 1, 2 and 3.

Let us also study the asymptotic behavior of cartesian products of MRD codes. Let \( C \) be an \((n, k, d_k)\) MRD code over GF\((q^m)\) \((n \leq m)\), and let \( C^l \) be the code obtained by \( l \) cartesian products of \( C \) with length \( n' = nl \), dimension \( k' = kl \), and minimum rank distance \( d'_k = d_k = n - k + 1 \). Clearly, we have \( k' = n' - l(d'_k - 1) \). Thus, \( \lim_{n \to \infty} \frac{k'}{n'} = 1 - l\delta \) for \( 0 \leq \delta \leq l^{-1} \). Let us define \( b = \lim_{n \to \infty} \frac{n'}{m'} \), then we have \( l^{-1} \leq b^{-1} \). Hence, \( C^l \) reaches or exceeds the Gilbert-Varshamov bound if and only if \( \delta \leq \frac{b+1-l}{b} \). We can show that when \( l \geq b + 1 \), \( C^l \) does not attain the Gilbert-Varshamov bound and hence is not asymptotically maximal for any \( \delta > 0 \). This confirms the result in Proposition 2.

IV. COVERING RADIUS

Let \( C \) be an \((n, k)\) code over GF\((q^m)\). The covering radius in the rank metric \( r(C) \) of this code is defined in [11] similarly to the covering radius in the Hamming metric. It is the smallest integer \( r \) such that all vectors in the space GF\((q^m)^n \) are within rank distance \( r \) of some codeword. The covering radius is an important geometric property of a code: it is a measure of the maximum distortion if the code is used for data compression, and is the maximum weight of a correctable random error if the code is used for error correction [20].

A. General properties of the covering radius

From the definition of the covering radius, it is clear that for any code \( C \), \( r(C) \leq m \). Also, if we further assume that \( C \) is linear, \( r(C) \) is bounded by \( n - k \) [11]. Similarly to the Hamming covering radius, the rank covering radius of a maximal code \( C \) satisfies \( r(C) \leq d_k(C) - 1 \), where \( d_k(C) \) is the minimum rank distance of \( C \). Combining this result with the Singleton bound for the rank metric in (3), we obtain that for any (linear or nonlinear) maximal code \( C \),

\[
 r(C) \leq \min \{ n - k, \left\lfloor \frac{m}{n}(n - k) \right\rfloor \}. \tag{11}
\]

Note that the bound in (11) is not applicable to general codes with the rank metric. A trivial counter example is given by the \((n, 1)\) repetition code with length \( n \geq m + 1 \), which has covering radius \( n-1 \) [11] that exceeds the bound in (11).

B. Covering radius of MRD codes

If we assume that \( n \leq m \), then (11) becomes \( r(C) \leq n - k \). In the following, we show that generalized Gabidulin codes have the maximum covering radius of \( n - k \). The proof follows the same arguments used in the derivation of the Hamming covering radius of Reed-Solomon codes [20].

**Definition 3:** Let \( C_1 \subset C_2 \) be two codes. We denote by \( m(C_2, C_1) \) [and \( M(C_2, C_1) \)] the weight of the translate-leader of least [greatest] nonzero rank weight among the translates of \( C_1 \) by elements of \( C_2 \), i.e.,

\[
 m(C_2, C_1) = \min_{x \in C_2 - C_1} \{ w(x + c) \mid c \in C_1 \}, \tag{12}
\]

\[
 M(C_2, C_1) = \max_{x \in C_2} \min_{c \in C_1} \{ w(x + c) \}. \tag{13}
\]

When \( C_1 \) and \( C_2 \) are linear, these are the minimum nonzero and maximum weights of cosets of \( C_2 \) mod \( C_1 \).

We remark that \( M(C_2, C_1) \geq M(C_2, C_1) \), and that if \( C_2 \subset C_3 \), then \( M(C_3, C_1) \geq M(C_2, C_1) \).

**Lemma 1 (Supercode Lemma):** Let \( C_1 \) and \( C_2 \) be two linear codes such that \( C_1 \subset C_2 \). Then \( r(C_1) \geq M(C_2, C_1) \geq m(C_2, C_1) \). Also,

\[
 r(C_1) \geq \min_{x \in C_2 - C_1} \{ w(x) \}. 
\]
Using Lemma 1, we can show that generalized Gabidulin codes have maximal covering radius.

**Proposition 5:** An \((n, k, \mathbb{F}_q)\) generalized Gabidulin code over \(\mathbb{F}_q^n\) \((m \geq n)\) has covering radius \(d_k - 1 = n - k\).

A similar argument can be used to bound the covering radius of the cartesian products of generalized Gabidulin codes.

**Corollary 1:** Let \(C\) be an \((n, k, d_k)\) generalized Gabidulin code \((s \leq m)\), and let \(C'\) be the \((n', k', d_k')\) code obtained by \(l\) cartesian products of \(C\). Then, \(\forall l \geq 1\), the rank covering radius of \(C'\) satisfies \(r(C') \geq d_k' - 1\).

Note that we do not have an equality as in Corollary 5 since cartesian products of MRD codes are not necessarily maximal, as stated in Proposition 2. However, when \(n = m\), \(C'\) is an MRD code, therefore its covering radius also satisfies \(r(C') \leq d_k(C') - 1\). This leads to the following result.

**Corollary 2:** Let \(C\) be an \((m, k, d_k)\) generalized Gabidulin code over \(\mathbb{F}_q^m\), and let \(C'\) be the \((n', k', d_k')\) code obtained by \(l\) cartesian products of \(C\). Then \(r(C') = d_k' - 1 = \frac{m}{n}(n' - k')\).

In summary, we conclude that all three subclasses of MRD codes known so far have maximal covering radius. Note that, the situation is different for codes with the Hamming metric: It is known that some MDS codes — such as the repetition code — do not have maximal covering radius.

**V. Rank weight distribution**

All public-key cryptosystems based on error-control codes encrypt the plaintext by first encoding it using the public code and then adding an error vector of weight \(t\) (see, for example, [4], [7] for details). Thus, a brute force decoding attack [7] can be used to break these cryptosystems: first guess the error vector of weight \(t\), and then subtract the guessed error vector and invert the encoding process; the system is broken if the guessed error vector is the error vector used in the encryption operation, otherwise repeat with a different guess. Clearly, the work factor of such a decoding attack is proportional to the number of the vectors with weight \(t\), \(N_q^m(n, t)\). For codes with the rank metric, it is given in (5). For codes with the Hamming metric, it is given by \((q^m - 1)^t(n)\). It can be shown that for \(0 < r < n\) the number of the vectors with the rank weight \(r\) is much greater than the number of the vectors with the Hamming weight \(r\). The numbers of vectors with length 32 and rank and Hamming weight \(r\) \((0 \leq r \leq 32)\), respectively, over \(\mathbb{F}_2(2^{32})\) are compared in Figure 4. Thus, the public-key cryptosystems based on rank metric codes are more secure against this brute force decoding attack. This partially explains why the GPT cryptosystem and its variants are quite secure against decoding attacks.

**References**


Fig. 4. The number of vectors with length 32 and weight \(r\) over \(\mathbb{F}_2(2^{32})\)